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Bethe Ansatz Equations for the Broken $\mathbf{Z}_N$-Symmetric Model

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We obtain the Bethe ansatz equations for the broken $\mathbf{Z}_N$-symmetric model by constructing a functional relation of the transfer matrix of L-operators. This model is an elliptic off-critical extension of the Fateev–Zamolodchikov model. We calculate the free energy of this model on the basis of the string hypothesis.

KEY WORDS: $\mathbf{Z}_2$-symmetry; broken $\mathbf{Z}_N$-symmetric model; transfer matrices; functional relation; Bethe ansatz; string hypothesis; free energy.

1. INTRODUCTION

In the two-dimensional solvable lattice models with Ising-like edge interaction, the star–triangle relation

$$\begin{aligned} &\rho W(a, b \mid v, w)\, \bar{W}(a, c \mid u, w)\, W(b, c \mid u, v) \\ &\quad = \sum_d \bar{W}(a, d \mid u, v)\, W(d, b \mid u, w)\, \bar{W}(d, c \mid v, w) \qquad (1.1) \\ &\rho = \rho(u, v, w) \quad \text{independent of } a, b, \text{ and } c \end{aligned}$$

plays a central role. In (1.1), the summation on d is taken over all local states. These are the relations among the two Boltzmann weights $W(a, b \mid u, v)$ and $\bar{W}(a, b \mid u, v)$. They live on the edges in two different directions of the two-dimensional planar lattice. The local state variables a and b live on the sites. We denote the spectral parameters by u and v. (See Fig. 1.)

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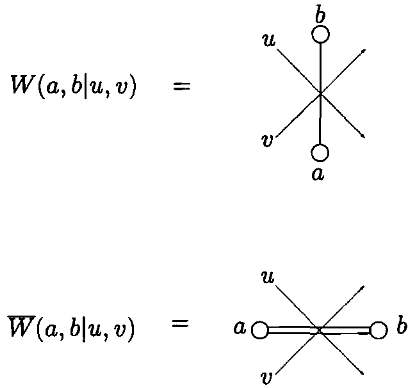


Fig. 1. Graphical representation of $W(a, b \mid u, v)$ and $\bar{W}(a, b \mid u, v)$.

Since Fateev and Zamolodchikov[1] obtained an N-state generalization of the critical Ising model as a solution of the star–triangle relation (STR), there have been two different off-critical extensions of this model. One is the chiral Potts model[2] and the other is the broken $\mathbf{Z}_N$-symmetric model. Both are Ising-type edge interaction models. The STR for the chiral Potts model was proved in refs. 3 and 4. Though this model is still under investigation,[5-10] the lack of a difference-variable parametrization in this model causes difficulties in analysis. Kashiwara and Miwa[11] proposed the broken $\mathbf{Z}_N$-symmetric model, and Hasegawa and Yamada[12] proved the STR for this model. Unfortunately the proof in ref. 11 was wrong because of the incorrectness of the "ICU lemma" in their paper.

In this paper, we study the eigenvalues of the transfer matrix $\Phi(u, v, w)$ of the broken $\mathbf{Z}_N$-symmetric model,

$$\Phi(u, v, w)_{a_0 a_1 \cdots a_{M-1}}^{b_0 b_1 \cdots b_{M-1}} = \prod_{j=0}^{M-1} \bar{W}(b_j, a_j \mid v-w)\, W(a_j, b_{j+1} \mid u-w) \quad (1.2)$$

and calculate the free energy of this model. (See Fig. 3.)

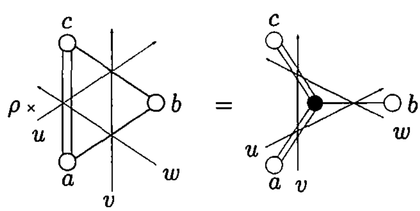


Fig. 2. Graphical representation of the star–triangle relation.

$$\Phi(u, v, w)^{b_0 b_1 \cdots b_{M-1}}_{a_0 a_1 \cdots a_{M-1}} =$$

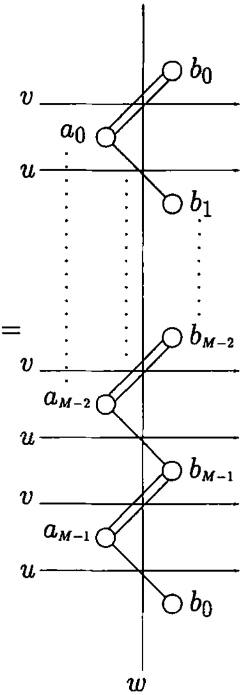


Fig. 3. Graphical representation of $\Phi(u, v, w)$.

The local state variables take their values in $\mathbf{Z}/N\mathbf{Z}$. Throughout the paper, we deal with the case of N odd, $N = 2n+1$. The $\mathbf{Z}_2$-symmetry of the Boltzmann weights

$$W(a, b \mid u) = W(N-a, N-b \mid u), \qquad \bar{W}(a, b \mid u) = \bar{W}(N-a, N-b \mid u) \tag{1.3}$$

ensures that the eigenvalue $r = \pm 1$ of the spin reversal operator $\mathscr{R}$ is a good quantum number, where $\mathscr{R} \in \mathrm{End}((\mathbf{C}^N)^{\otimes M})$ is defined by

$$\mathscr{R} = \overbrace{R \otimes R \otimes \cdots \otimes R}^{M \text{ times}}, \qquad R v_j^{(N)} = v_{N-j}^{(N)} \tag{1.4}$$

which satisfies $\mathscr{R}^2 = 1$. The vectors $v_j^{(N)}$ $(j \in \boldsymbol{Z}/N\mathbf{Z})$ constitute an orthonormal basis in $\mathbf{C}^N$. In the homogeneous case $u = v$, we show first that any eigenvalue $\varphi(u)$ of $\Phi(u) = \Phi(u, u, 0)$ can be written as

$$\varphi(u) = \left(\frac{p(0)\,p(\lambda)}{p(u)\,p(\lambda-u)}\right)^{M} \prod_{j=1}^{2nM} \frac{\theta_1(u-u_j \mid \tau/2)}{\theta_1(u_j \mid \tau/2)} \tag{1.5}$$

$$p(u) = \prod_{j=1}^{n} \theta_2(u-(2j-1)\eta \mid \tau/2), \qquad \eta = \frac{n}{N}, \quad \lambda = \frac{1}{2} - \eta \tag{1.6}$$

See Appendix A for the notation of the theta functions. The zeros $\{u_1,..., u_{2nM}\}$ of $\varphi(u)$ are described as follows:

$$\left(\frac{\theta_1(v_k+\lambda/2 \mid \tau/2)}{\theta_1(v_k-\lambda/2 \mid \tau/2)}\right)^{2M} = (-1)^{M+1} \prod_{j=1}^{2nM} \frac{\theta_1(v_k-v_j+\eta \mid \tau/2)}{\theta_1(v_k-v_j-\eta \mid \tau/2)} \tag{1.7}$$

$$v_k = u_k - \frac{\lambda}{2} \qquad \text{for} \quad k = 1,..., 2nM \tag{1.8}$$

$$\sum_{j=1}^{2nM} v_j \equiv \frac{1-r}{4} \bmod \left(\mathbf{Z} \oplus \frac{\tau}{2}\mathbf{Z}\right) \tag{1.9}$$

We call Eq. (1.7) the Bethe ansatz equation. The condition (1.9) follows from the double periodicity of $\varphi(u)$ discussed in Section 4. We obtain the

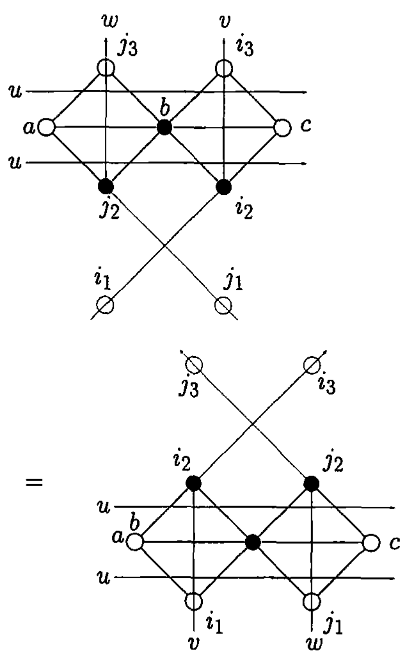


Fig. 4. Graphical representation of (1.10).

Bethe ansatz equations above through a functional relation (1.13) for the transfer matrix of L-operators. These L-operators $L(u) \in \mathrm{End}(\mathbf{C}^N \otimes \mathbf{C}^2)$ were originally constructed by Sklyanin[13,14] as a solution to the relation (Fig. 4)

$$L^{01}(u-v)\, L^{02}(u-w)\, R^{2}_{8\mathrm{V}}(v-w)$$
$$= R^{12}_{8\mathrm{V}}(v-w)\, L^{02}(u-w)\, L^{01}(u-v) \qquad \text{on} \quad \mathbf{C}^N \otimes \mathbf{C}^2 \otimes \mathbf{C}^2 \tag{1.10}$$

where the upper indices 0, 1, and 2 mean that $L^{ij}(u)$ acts only on the ith and jth components of $\mathbf{C}^N \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ and as identity on the other components. We denote the R-matrix of the eight-vertex model by $R_{8\mathrm{V}}(u)$.[15,16] We consider the transfer matrix $\mathscr{L}(u)$ of these L-operators,

$$\mathscr{L}(u) = \mathrm{tr}_{\mathbf{C}^2}(L^{0M}(u)\, L^{1M}(u) \cdots L^{M-1M}(u)) \tag{1.11}$$

$$L^{0M}(u)\, L^{1M}(u) \cdots L^{M-1M}(u) \in \mathrm{End}(\overbrace{\mathbf{C}^N \otimes \cdots \otimes \mathbf{C}^N}^{M \text{ times}} \otimes \mathbf{C}^2) \tag{1.12}$$

We derive the functional relation

$$\mathscr{L}(\lambda - u - 1/4)\, \Phi(u) = C(u)^M (f(u)^M \Phi(u-\eta) + (-f(\lambda-u))^M \Phi(u+\eta)) \tag{1.13}$$

$$f(u) = \theta_1(2u \mid \tau) \frac{\theta_1(u+\eta \mid \tau/2)}{\theta_2(u \mid \tau/2)} \tag{1.14}$$

$$C(u) = [\theta_2\theta_3\theta_4](0 \mid \tau) \prod_{j=1}^{n} \frac{\theta_1(u-2(j-1)\eta \mid \tau/2)\, \theta_2(u+(2j-1)\eta \mid \tau/2)}{\theta_1(u+2j\eta \mid \tau/2)\, \theta_2(u-(2j-1)\eta \mid \tau/2)} \tag{1.15}$$

by the method which Baxter employed to solve the eight-vertex model.[15-17] The correspnding functional relations in the chiral Potts model and in the RSOS model associated with the eight-vertex model were obtained in refs. 18 and 19, respectively.

We calculate the free energy of the broken $\mathbf{Z}_N$-symmetric model under the hypothesis that the solution of the Bethe ansatz equations corresponding to the ground state consists of "strings of length $N-1$." We show that, in the infinite lattice limit, the centers of these strings are distributed on the imaginary axis with the density $\rho(w)$,

$$\rho(w) = 2N[\theta_2\theta_3](0 \mid N\tau) \frac{\theta_3(2\sqrt{-1}\, Nw \mid N\tau)}{\theta_2(2\sqrt{-1}\, Nw \mid N\tau)}, \qquad -\frac{\kappa}{4} \leqslant w < \frac{\kappa}{4} \tag{1.16}$$

where $\tau=\sqrt{-1}\,\kappa$, and the free energy per site is

$$F(u)=-\sum_{l=1}^{\infty}\left(\left\{\sinh\left(\frac{2\pi l}{\kappa}u\right)\sinh\left[\frac{2\pi l}{\kappa}\left(\frac{1}{2N}-u\right)\right]\sinh\left(\frac{2\pi l}{N\kappa}n\right)\right\}\right.$$
$$\left.\times\left[l\cosh\left(\frac{\pi l}{\kappa}\right)\cosh^2\left(\frac{\pi l}{N\kappa}\right)\right]^{-1}\right) \tag{1.17}$$

The last expression agrees with the result of Jimbo *et al.*[20] obtained by the use of the inversion trick, and in the trigonometric limit of $\kappa\to\infty$ it recovers the results of Fateev and Zamolodchikov[1] and Albertini.[10]

The organization of this paper is as follows. In Section 2, we review necessary facts about the R-matrix of the eight-vertex model, Sklyanin's L-operator, and the broken $\mathbf{Z}_N$-symmetric model. We derive the functional relation (1.13) in Section 3. After showing commutation relations among $\Phi(u)$, $\mathscr{L}(v)$, and $\mathscr{R}$, we obtain the Bethe ansatz equations in Section 4. We calculate the free energy of the model under the string hypothesis in Section 5. Finally, in Section 6, we conclude with a brief discussion. We fix the notation and list the formulas for theta functions in Appendix A. Miscellaneous properties of the Boltzmann weights are summarized in Appendix B. We devote Appendix C to the proof of the commutativity between Φ and $\mathscr{L}$.

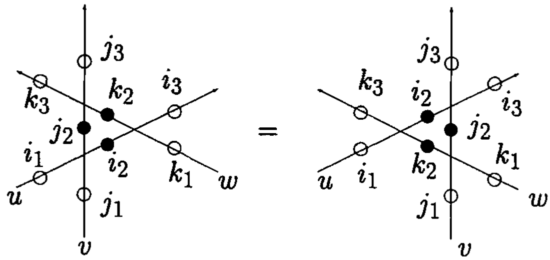
2. REVIEW OF THE BROKEN $\mathbf{Z}_N$ SYMMETRIC MODEL

We fix the notation for matrices. We denote the vector ${}^t(0,\dots,\overset{j}{\check{1}},\dots,0)$ in $\mathbf{C}^m$ by $v_j^{(m)}$, $j=0,1,\dots,m-1$, and the matrix elements of $A\in\mathrm{End}(\mathbf{C}^{m_1}\otimes\mathbf{C}^{m_2}\otimes\cdots\otimes\mathbf{C}^{m_l})$ by

$$Av_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_l}=\sum_{j_1=0}^{m_1-1}\sum_{j_2=0}^{m_2-1}\cdots\sum_{j_l=0}^{m_l-1}v_{j_1}\otimes v_{j_2}\otimes\cdots\otimes v_{j_l}A_{i_1i_2\cdots i_l}^{j_1j_2\cdots j_l}$$

Sklyanin[13,14] constructed the L-operators $L(u)\in\mathrm{End}(\mathbf{C}^N\otimes\mathbf{C}^2)$ for the eight-vertex model satisfying (1.10). The R-matrix of the eight-vertex model $R_{8\mathrm{V}}(u)$ is a solution of the Yang–Baxter equation (Fig. 5),[15,16]

$$R_{8\mathrm{V}}^{01}(u-v)\,R_{8\mathrm{V}}^{02}(u-w)\,R_{8\mathrm{V}}^{12}(v-w)$$
$$=R_{8\mathrm{V}}^{12}(v-w)\,R_{8\mathrm{V}}^{02}(u-w)\,R_{8\mathrm{V}}^{01}(u-v)\qquad\text{on}\quad\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathbf{C}^2 \tag{2.1}$$



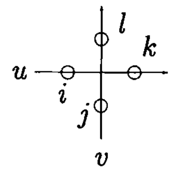
$R^{kl}_{ij}(u-v) \quad =$ 

Fig. 5. Graphical representation of the Yang–Baxter equation.

Its nonzero matrix elements are

$$R^{00}_{00}(u)=R^{11}_{11}(u)=[\theta_2\theta_3](\eta)[\theta_2\theta_3](u)[\theta_1\theta_4](u+\eta)$$
$$R^{01}_{01}(u)=R^{10}_{10}(u)=[\theta_2\theta_3](\eta)[\theta_1\theta_4](u)[\theta_2\theta_3](u+\eta)$$
$$R^{01}_{10}(u)=R^{10}_{01}(u)=[\theta_1\theta_4](\eta)[\theta_2\theta_3](u)[\theta_2\theta_3](u+\eta)$$
$$R^{00}_{11}(u)=R^{11}_{00}(u)=[\theta_1\theta_4](\eta)[\theta_1\theta_4](u)[\theta_1\theta_4](u+\eta)$$

The other elements not specified above are all zero. Here we denote $\theta_1(u)\,\theta_4(u)$ by $[\theta_1\theta_4](u)$ for short. We usually suppress the elliptic modulus τ. When $\eta=n/N$, $N=2n+1$, the L-operators $L(u)$ in (1.10) have a "cyclic" representation. In this representation, $L(u)$ factorizes elementwise as

$$L^{bj}_{ai}(u)=K_{ia}^{\ \ b}(u)\,K^{jb}_{\ \ a}(u) \tag{2.2}$$

where

$$K^{ja}_{\ \ a+\sigma}(u)=(-1)^{j(1+\sigma)/2}[\theta_{1+j}\theta_{4-j}](u-\sigma a\eta+\tfrac{1}{4}) \tag{2.3}$$

$$K_{ja}^{\ \ b}(u)=G_a^{-1}G_b^{-1}K^{ja}_{\ \ b}(u), \qquad G_a=\left(\frac{\theta_4(2a\eta)}{\theta_4(0)}\right)^{1/2} \tag{2.4}$$

for $j=0, 1$, $a, b=0, 1,..., N-1$, and $\sigma=\pm 1$. (See Fig. 6.)

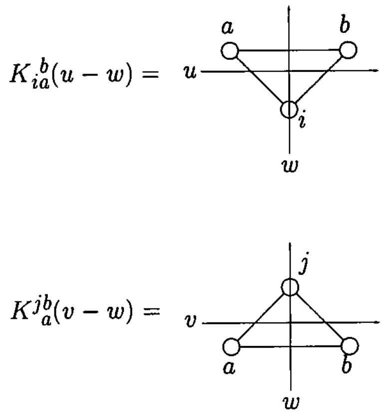


Fig. 6. Graphical representations of $K_{ia}^{\,b}(u-w)$ and $K^{jb}_{\;\;a}(v-w)$.

The factors $K_{ia}^{\,b}(u)$ and $K^{jb}_{\;\;a}(u)$ are zero unless $|a-b|=1$. We can identify these $K(u)$'s as the intertwining vectors appearing in the vertex–face correspondence.[21–26] Even in the Fateev–Zamolodchikov model these $K(u)$'s are different from the three-spin object V's in refs. 2 and 5 by definition. Their V's are defined by the Fourier-transformed images of the product of two Boltzmann weights. These two objects, however, should have an intimate relationship, because the transfer matrix $\mathscr{L}$ in the Fateev–Zamolodchikov model is also constructed from V's.[5]

Under the $\mathbf{Z}_2$-transformation which sends a to $N-a$, they change as

$$K^{jN-a}_{\;\;N-b}(u)=(-1)^{j(n-1)+1}\,K^{ja}_{\;\;b}(u), \qquad K_{jN-b}^{\;N-a}(u)=(-1)^{j(n-1)+1}\,K_{jb}^{\;a}(u) \tag{2.5}$$

They satisfy the unitarity relations[27] Fig. 7)

$$\sum_{j=0}^{1} G_a K_{ja}^{\,b}(u+\lambda)\,K^{jb}_{\;\;c}(u)=\delta_{ac}[\theta_2\theta_3\theta_4](0)\,G_b\theta_2(2u) \tag{2.6}$$

$$\sum_{a=0}^{N-1} G_a K_{ia}^{\,b}(u+\lambda)\,K^{jb}_{\;\;a}(u)=\delta_{ij}[\theta_2\theta_3\theta_4](0)\,G_b\theta_2(2u) \tag{2.7}$$

$$\sum_{b=0}^{N-1} G_b K_{ia}^{\,b}(u)\,K^{jb}_{\;\;a}(u+\lambda)=\delta_{ij}[\theta_2\theta_3\theta_4](0)\,G_a\theta_2(2u) \tag{2.8}$$

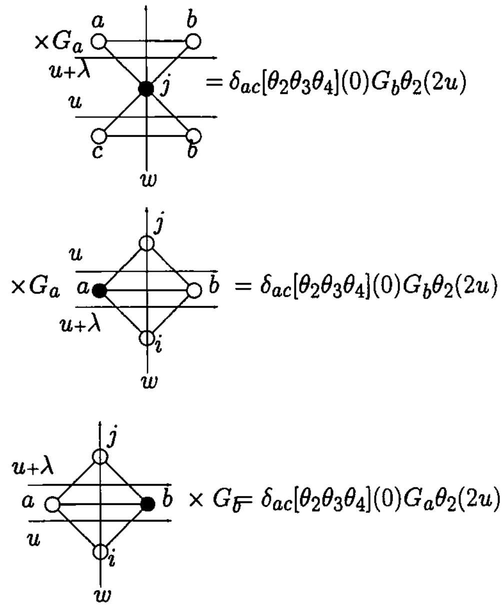


Fig. 7. Graphical representations of unitary relations between the K's.

In ref. 12 we determined the Boltzmann weights W and $\bar{W}$ of the broken $\mathbf{Z}_N$-symmetric model by the relations (Figs. 8 and 9)

$$\begin{aligned} W(a, b \mid u, v) &\sum_{j=0}^{1} K^{jc}_{a}(u-w)\, K^{d}_{jb}(v-w) \\ &= \sum_{j=0}^{1} K^{jc}_{a}(v-w)\, K^{d}_{jb}(v-w)\, W(c, d \mid u, v) \end{aligned} \tag{2.9}$$

$$\begin{aligned} \sum_{b=0}^{N-1} &\bar{W}(a, b \mid u, v)\, K^{c}_{ib}(u-w)\, K^{jc}_{b}(v-w) \\ &= \sum_{b=0}^{N-1} K^{b}_{ia}(v-w)\, K^{jb}_{a}(u-w)\, \bar{W}(b, c \mid u, v) \end{aligned} \tag{2.10}$$

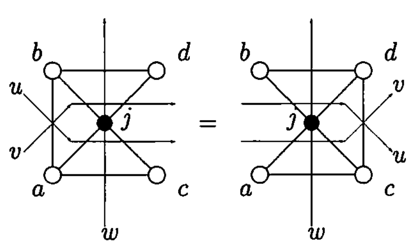


Fig. 8. Graphical representation of (2.9).

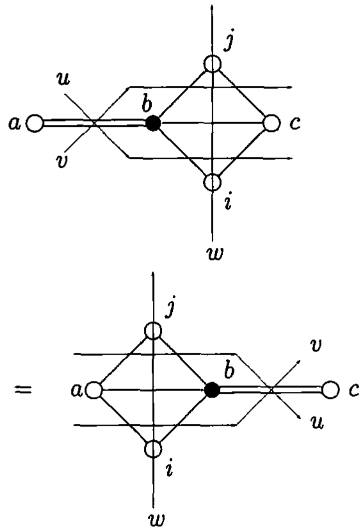


Fig. 9. Graphical representation of (2.10).

From the above relations and (2.5), we have the $\mathbf{Z}_2$-symmetry

$$\begin{aligned} W(a, b \mid u, v) &= W(N-a, N-b \mid u, v) \\ \bar{W}(a, b \mid u, v) &= \bar{W}(N-a, N-b \mid u, v) \end{aligned} \tag{2.11}$$

Equation (2.9) implies that $W(a, b \mid u, v) = W(a, b \mid u-v)$ and that

$$\begin{aligned} \frac{W(a+1, b+1 \mid u)}{W(a, b \mid u)} &= \frac{\theta_3(u+(a+b+1)\eta)}{\theta_3(u-(a+b+1)\eta)} \\ \frac{W(a+1, b-1 \mid u)}{W(a, b \mid u)} &= \frac{\theta_2(u+(a-b+1)\eta)}{\theta_2(u-(a-b+1)\eta)} \end{aligned} \tag{2.12}$$

The crossing symmetry

$$\bar{W}(a, b \mid u) = G_a G_b W(a, b \mid \lambda - u) \tag{2.13}$$

holds in this model. Hasegawa[27] proved the crossing symmetry only from (2.9) and the unitarity relations (2.7) and (2.8). We thus have

$$\begin{aligned}\frac{\bar{W}(a+1,b+1\mid u)}{\bar{W}(a,b\mid u)}&=\frac{G_{a+1}G_{b+1}}{G_aG_b}\frac{\theta_4(u-(a+b)\eta)}{\theta_4(u+(a+b+2)\eta}\\ \frac{\bar{W}(a+1,b-1\mid u)}{\bar{W}(a,b\mid u)}&=\frac{G_{a+1}G_{b-1}}{G_aG_b}\frac{\theta_1(u-(a-b)\eta)}{\theta_1(u+(a-b+2)\eta}\end{aligned} \tag{2.14}$$

without directly solving (2.10). We can see from (2.12) and (2.14) that the Boltzmann weights satisfy the reflection symmetry

$$W(a,b\mid u)=W(b,a\mid u), \qquad \bar{W}(a,b\mid u)=\bar{W}(b,a\mid u)$$

Defining $T_k^{(+)}(\alpha\mid u)$ and $T_k^{(-)}(\alpha\mid u)$ by

$$T_k^{(+)}(\alpha\mid u)=\prod_{j=1}^{\alpha}\frac{\theta_k(u+(2j-1)\eta)}{\theta_k(u-(2j-1)\eta)}, \qquad T_k^{(-)}(\alpha\mid u)=T_k^{(+)}(\alpha\mid\lambda-u) \tag{2.15}$$

respectively, we find the solutions to the recursion relations (2.12) and (2.14) under the normalization $W(0,0\mid u)=\bar{W}(0,0\mid u)=1$:

$$\begin{aligned}W(2a,2b\mid u)&=T_2^{(+)}(a-b\mid u)\,T_3^{(+)}(a+b\mid u)\\ \bar{W}(2a,2b\mid u)&=G_{2a}G_{2b}T_2^{(-)}(a-b\mid u)\,T_3^{(-)}(a+b\mid u)\end{aligned}$$

Here all local state variables are to be read modulo N. See Appendix B for details. Hasegawa and Yamada[12] established the star–triangle relation (STR) in the broken Z_N-symmetric model,

$$\begin{aligned}&\rho W(a,b\mid v-w)\,\bar{W}(a,c\mid u)\,W(b,c\mid u-v)\\ &\quad=\sum_{d=0}^{N-1}\bar{W}(a,d\mid u-v)\,W(d,b\mid u-w)\,\bar{W}(d,c\mid v-w)\end{aligned} \tag{2.16}$$

where ρ is a scalar function independent of a, b, and c.

3. FUNCTIONAL RELATION

In this section, we consider the transfer matrix of the L-operators and construct a functional relation for it. In the course of the calculation, we utilize the factorization property of L into K's, (2.2). We define a 2-by-2 matrix $L(a,b\mid u,v,w)$ by

$$L(a,b\mid u,v,w)=\begin{pmatrix}K_{0a}^{\ b}(u-w)\,K^{0b}_{\ \ a}(v-w) & K_{0a}^{\ b}(u-w)\,K^{1b}_{\ \ a}(v-w)\\ K_{1a}^{\ b}(u-w)\,K^{0b}_{\ \ a}(v-w) & K_{1a}^{\ b}(u-w)\,K^{1b}_{\ \ a}(v-w)\end{pmatrix}$$

$$\mathcal{L}(u,v,w)_{a_0 a_1 \cdots a_{M-1}}^{b_0 b_1 \cdots b_{M-1}} =$$

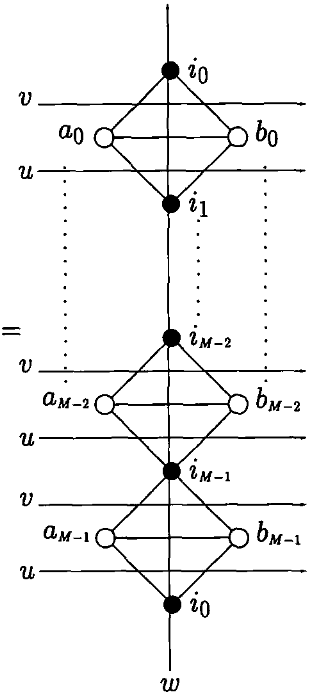


Fig. 10. Graphical representations of $\mathscr{L}(u, v, w)$.

Then the transfer matrix $\mathscr{L}(u, v, w)$ of L-operators on the lattice of width M with the periodic boundary condition is (Fig. 10)

$$\begin{aligned}
&\mathscr{L}(u, v, w)_{a_0 a_1 \cdots a_{M-1}}^{b_0 b_1 \cdots b_{M-1}} \\
&\quad = \operatorname{tr}(L(a_0, b_0 \mid u, v, w)\, L(a_1, b_1 \mid u, v, w) \cdots L(a_{M-1}, b_{M-1} \mid u, v, w)) \\
&\quad = \sum_{i_0, \ldots, i_{M-1}} \prod_{j=0}^{M-1} K_{i_{j+1}}{}_{a_j}^{b_j}(u-w)\, K^{i_j}{}_{a_j}^{b_j}(v-w)
\end{aligned} \tag{3.1}$$

The final goal of this section is to establish the functional relation

$$\begin{aligned}
&\mathscr{L}(\lambda - u, \lambda - v, w + 1/4)\, \Phi(u, v, w) \\
&\quad = C(u, v, w)^M \begin{pmatrix} f(u, v, w)^M \, \Phi(u, v, w + \eta) \\ \qquad + (-f(\lambda - v, \lambda - u, -w))^M \, \Phi(u, v, w - \eta) \end{pmatrix}
\end{aligned} \tag{3.2}$$

where we define $C(u, v, w)$ and $f(u, v, w)$ by

$$C(u, v, w) = [\theta_2\theta_3\theta_4](0)[T_2^{(+)}T_3^{(+)}](n \mid u-w)[T_2^{(-)}T_3^{(-)}](n \mid v-w) \quad (3.3)$$

$$f(u, v, w) = \theta_1(2u-2w)\frac{[\theta_1\theta_4](v-w+\eta)}{[\theta_2\theta_3](u-w)} \quad (3.4)$$

This functional relation reduces to (1.13)–(1.15) in the homogeneous case, $u = v$. We achieve this goal by the method à la Baxter,[15–17] which states the following: Suppose that we can find $\mathbf{C}$-valued functions $\phi_j^{(\nu)}(b \mid u, v, w)$ ($\nu = 0, 1, 2, 3$ and $b \in \mathbf{Z}/N\mathbf{Z}$) and matrices $P_j \in \mathrm{End}(\mathbf{C}^2)$ for $j \in \mathbf{Z}/M\mathbf{Z}$ which satisfy

$$P_j^{-1} \sum_{b=0}^{N-1} \phi_j^{(0)}(b \mid u, v, w)\, L(a, b \mid u, v, w)\, P_{j+1}$$
$$= \begin{pmatrix} \phi_j^{(1)}(a \mid u, v, w) & \phi_j^{(3)}(a \mid u, v, w) \\ 0 & \phi_j^{(2)}(a \mid u, v, w) \end{pmatrix} \quad \text{for } a \in \mathbf{Z}/N\mathbf{Z} \text{ and } j \in \mathbf{Z}/M\mathbf{Z} \quad (3.5)$$

Then we have

$$\sum_{b_0, \ldots, b_{M-1}} \phi_0^{(0)}(b_0 \mid u, v, w)\, \phi_1^{(0)}(b_1 \mid u, v, w)$$
$$\times \cdots \phi_{M-1}^{(0)}(b_{M-1} \mid u, v, w)\, \mathscr{L}(u, v, w)_{a_0 a_1 \cdots a_{M-1}}^{b_0 b_1 \cdots b_{M-1}}$$
$$= \prod_{j=0}^{M-1} \phi_j^{(1)}(a_j \mid u, v, w) + \prod_{j=0}^{M-1} \phi_j^{(2)}(a_j \mid u, v, w) \quad (3.6)$$

Defining vectors $\psi^{(\nu)}(u, v, w) \in (\mathbf{C}^N)^{\otimes M}$ by

$$\psi^{(\nu)}(u, v, w)_{a_0 a_1 \cdots a_{M-1}} = \prod_{j=0}^{M-1} \phi_j^{(\nu)}(a_j \mid u, v, w) \quad (3.7)$$

we can write (3.6) as

$$\mathscr{L}(u, v, w)\, \psi^{(0)}(u, v, w) = \psi^{(1)}(u, v, w) + \psi^{(2)}(u, v, w) \quad (3.8)$$

In the following, we will find a family of solutions to (3.5)

$$\phi_j^{(\nu)}(b \mid u, v, w) = \phi^{(\nu)}(c_j, b, c_{j+1} \mid u, v, w) \qquad (\nu = 0, 1, 2, 3 \text{ and } b \in \mathbf{Z}/N\mathbf{Z})$$
$$P_j = P(c_j)$$

labeled by $\{(c_0, c_1, \ldots, c_{M-1}) \mid c_j \in \mathbf{Z}/N\mathbf{Z} \text{ for } j \in \mathbf{Z}/M\mathbf{Z}\}$. This gives rise to N^M vectors $\psi^{(\nu)}$ labeled as $\psi^{(\nu)}(u, v, w)^{c_0, c_1, \ldots, c_{M-1}}$. We will also prove that

ψ's are proportional to the row vectors of the diagonal-to-diagonal transfer matrix Φ of the broken $\mathbf{Z}_N$-symmetric model,

$$\psi^{(0)}(\lambda-u,\lambda-v,w+1/4)^{c_0c_1\cdots c_{M-1}}_{b_0b_1\cdots b_{M-1}} = \Phi(u,v,w)^{c_0c_1\cdots c_{M-1}}_{b_0b_1\cdots b_{M-1}} \tag{3.9}$$

$$\psi^{(1)}(\lambda-u,\lambda-v,w+1/4)^{c_0c_1\cdots c_{M-1}}_{b_0b_1\cdots b_{M-1}} = (C(u,v,w)\,f(u,v,w))^M\,\Phi(u,v,w+\eta)^{c_0c_1\cdots c_{M-1}}_{b_0b_1\cdots b_{M-1}} \tag{3.10}$$

$$\psi^{(2)}(\lambda-u,\lambda-v,w+1/4)^{c_0c_1\cdots c_{M-1}}_{b_0b_1\cdots b_{M-1}} = (-C(u,v,w)\,f(\lambda-v,\lambda-u,-w))^M\,\Phi(u,v,w-\eta)^{c_0c_1\cdots c_{M-1}}_{b_0b_1\cdots b_{M-1}} \tag{3.11}$$

The results (3.8)–(3.11) together imply the functional relation (3.2).

Now we start to solve (3.5). We write the matrix elements of P_j as

$$P_j=\begin{pmatrix} p_j^{(0)} & p_j^{(2)} \\ p_j^{(1)} & p_j^{(3)} \end{pmatrix}$$

and its first column vector ${}^t(p_j^{(0)}, p_j^{(1)})$ as $\mathbf{p}_j$. Multiplying P_j to (3.5) from the left and taking its first column, we have

$$\sum_{b=0}^{N-1} \phi_j^{(0)}(b\mid u,v,w)\,L(a,b\mid u,v,w)\,\mathbf{p}_{j+1} = \phi_j^{(1)}(a\mid u,v,w)\,\mathbf{p}^j \qquad \text{for} \quad a\in\mathbf{Z}/N\mathbf{Z} \tag{3.12}$$

For later use, we define the functions $\Delta_{*(\pm)}$, $\Delta^*_{(\pm)}$, δ_*, and δ^* by

$$\begin{aligned}
\Delta_{*(\pm)}(\mathbf{p},a\mid u) &= p^{(0)}K_1{}^{a\pm1}_{\ \ a}(u) - p^{(1)}K_0{}^{a\pm1}_{\ \ a}(u)\\
\Delta^*_{(\pm)}(a,\mathbf{p}\mid u) &= K^{0a\pm1}_{\ \ \ a}(u)\,p^{(0)} + K^{1a\pm1}_{\ \ \ a}(u)\,p^{(1)}\\
\delta_*(a\mid u) &= K_0{}^{a-1}_{\ \ a}(u)\,K_1{}^{a+1}_{\ \ a}(u) - K_0{}^{a+1}_{\ \ a}(u)\,K_1{}^{a-1}_{\ \ a}(u)\\
\delta^*(a\mid u) &= K^{0a-1}_{\ \ \ a}(u)\,K^{1a+1}_{\ \ \ a}(u) - K^{0a+1}_{\ \ \ a}(u)\,K^{1a-1}_{\ \ \ a}(u)
\end{aligned}$$

Equations (3.12) constitute a system of $2N$ homogeneous linear equations in $\phi_j^{(0)}(a\mid u,v,w)$ and $\phi_j^{(1)}(a\mid u,v,w)$ for $a\in\mathbf{Z}/N\mathbf{Z}$. It has a nontrivial solution if and only if the determinant of its coefficient matrix vanishes. Demanding this condition, we obtain

$$\prod_{a=0}^{N-1} \Delta_{*(-)}(\mathbf{p}_j,a\mid u-w)\,\Delta^*_{(-)}(a,\mathbf{p}_{j+1}\mid v-w) + \prod_{a=0}^{N-1} \Delta_*(+)(\mathbf{p}_j,a\mid u-w)\,\Delta^*_{(+)}(a,\mathbf{p}_{j+1}\mid v-w)=0 \tag{3.13}$$

Later we find that Eq. (3.13) restricts $\mathbf{p}_j$ to a discrete set of values. When we parametrize $\mathbf{p}_j$ as

$$\mathbf{p}_j = \mathbf{p}(c_j), \qquad \mathbf{p}(c) = \begin{pmatrix} p^{(0)}(c) \\ p^{(1)}(c) \end{pmatrix} = \begin{pmatrix} [\theta_2\theta_3](c\eta) \\ -[\theta_1\theta_4](c\eta) \end{pmatrix} \tag{3.14}$$

and denote the dependence on $\mathbf{p}(c)$ simply by c and $U_\alpha = u + \alpha\eta$, we find that the Δ's and δ's become

$$\begin{aligned}
\Delta_{*(\pm)}(c, a \mid u - 1/4) &= \frac{[\theta_2\theta_3](0)}{G_a G_{a\pm 1}} \theta_2(U_{\pm c \mp a})\, \theta_3(U_{\mp c \mp a}) \\
\Delta^*_{(\pm)}(a, c \mid u - 1/4) &= [\theta_2\theta_3](0)\, \theta_1(U_{\pm a \mp c + 1})\, \theta_4(U_{\pm a \pm c + 1}) \\
\delta_*(a \mid u - 1/4) &= -\frac{[\theta_2\theta_3\theta_4](0)}{G_{a-1} G_{a+1}} \theta_1(2U_0) \\
\delta^*(a \mid u - 1/4) &= [\theta_2\theta_2\theta_4](0)\, G_a^2 \theta_1(2U_1)
\end{aligned}$$

Under the parametrization (3.14), the condition (3.13) holds if and only if either the c_j are all integers or all half-integers. We restrict outselves to the case that the c_j are all integers, because only in this case does the relation (3.8) give the functional relation (3.2). The system of equations (3.12) involves not all ϕ's, but only $\phi_j^{(0)}(a+1 \mid u, v, w)$, $\phi_j^{(0)}(a-1 \mid u, v, w)$, and $\phi_j^{(1)}(a \mid u, v, w)$. Expressing $\phi_j^{(0)}(a+1 \mid u, v, w)$ and $\phi_j^{(1)}(a \mid u, v, w)$ in terms of $\phi_j^{(0)}(a-1 \mid u, v, w)$, we obtain

$$\frac{\phi_j^{(0)}(a+1 \mid u, v, w)}{\phi_j^{(0)}(a-1 \mid u, v, w)} = -\frac{\Delta_{*(-)}(c_j, a \mid u - w)\, \Delta^*_{(-)}(a, c_{j+1} \mid v - w)}{\Delta_{*(+)}(c_j, a \mid u - w)\, \Delta^*_{(+)}(a, c_{j+1} \mid v - w)} \tag{3.15}$$

$$\frac{\phi_j^{(1)}(a \mid u, v, w)}{\phi_j^{(0)}(a-1 \mid u, v, w)} = \delta_*(a \mid u - w) \frac{\Delta^*_{(-)}(a, c_{j+1} \mid v - w)}{\Delta_{*(+)}(c_j, a \mid u - w)} \tag{3.16}$$

From (3.15) and (3.16), we can write $\phi_j^{(\nu)}(a \mid u, v, w)$ $(\nu = 0, 1)$ as

$$\phi_j^{(\nu)}(a \mid u, v, w) = \phi^{(\nu)}(c_j, a, c_{j+1} \mid u, v, w) \tag{3.17}$$

where the function $\phi^{(\nu)}(c, a, c' \mid u, v, w)$ is independent of j. Taking the determinant of both sides of (3.5), we find

$$\begin{aligned}
&\delta_*(c_j, a \mid u - w)\, \delta^*(a, c_{j+1} \mid v - w) \frac{\det(P_{j+1})}{\det(P_j)} \\
&= \frac{\phi^{(1)}(c_j, a, c_{j+1} \mid u, v, w)\, \phi_j^{(2)}(a \mid u, v, w)}{\phi^{(0)}(c_j, a-1, c_{j+1} \mid u, v, w)\, \phi^{(0)}(c_j, a+1, c_{j+1} \mid u, v, w)}
\end{aligned} \tag{3.18}$$

We set $\det(P_j)$ to unity without loss of generality. Then Eqs. (3.15), (3.16), and (3.18) give

$$\frac{\phi^{(2)}(c_j, a, c_{j+1} \mid u, v, w)}{\phi^{(0)}(c_j, a-1, c_{j+1} \mid u, v, w)} = -\delta^*(a \mid v-w) \frac{\Delta_{*(-)}(c_j, a \mid u-w)}{\Delta^*_{(+)}(a, c_{j+1} \mid v-w)} \tag{3.19}$$

where we write $\phi_j^{(2)}(a \mid u, v, w)$ as $\phi^{(2)}(c_j, a, c_{j+1} \mid u, v, w)$. The relations (3.15), (3.16), and (3.19) recursively determine $\phi_a^{(\nu)}$'s. We abbreviate $u-w+\alpha\eta$ and $v-w+\gamma\eta$ to A_α and B_γ, respectively. Comparing (3.15) with (2.12) and (2.14), we have

$$\frac{\phi^{(0)}(a, b+1, c \mid u, v, w+1/4)}{\phi^{(0)}(a, b-1, c \mid u, v, w+1/4)} = \frac{W(a, b+1 \mid A_0)\, \bar{W}(b+1, c \mid B_0)}{W(a, b-1 \mid A_0)\, \bar{W}(b-1, c \mid B_0)} \tag{3.20}$$

Hence we find that $\phi^{(0)}$ is a product of the two Boltzmann weights,

$$\phi^{(0)}(a, b, c \mid u, v, w+1/4) = W(a, b \mid A_0)\, \bar{W}(b, c \mid B_0) \tag{3.21}$$

From (3.15) and (3.16), we obtain

$$\frac{\phi^{(1)}(a, b+1, c \mid u, v, w+1/4)}{\phi^{(1)}(a, b-1, c \mid u, v, w+1/4)} = \frac{\phi^{(0)}(a, b+1, c \mid u-\eta, v-\eta, w+1/4)}{\phi^{(0)}(a, b-1, c \mid u-\eta, v-\eta, w+1/4)} \tag{3.22}$$

The same produce for $\phi^{(2)}$ yields

$$\frac{\phi^{(2)}(a, b+1, c \mid u, v, w+1/4)}{\phi^{(2)}(a, b-1, c \mid u, v, w+1/4)} = \frac{\phi^{(0)}(a, b+1, c \mid u+\eta, v+\eta, w+1/4)}{\phi^{(0)}(a, b-1, c \mid u+\eta, v+\eta, w+1/4)} \tag{3.23}$$

By (3.21)–(3.23), we can write $\phi^{(1)}$ and $\phi^{(2)}$ as

$$\phi^{(1)}(a, b, c \mid u, v, w+1/4) = f_{ac}(u, v, w)\, W(a, b \mid A_{-1})\, \bar{W}(b, c \mid B_{-1}) \tag{3.24}$$

$$\phi^{(2)}(a, b, c \mid u, v, w+1/4) = g_{ac}(u, v, w)\, W(a, b \mid A_{-1})\, \bar{W}(b, c \mid B_1) \tag{3.25}$$

where f_{ac} and g_{ac} are functions independent of b. Eqs. (3.16), (3.21), and (3.24) determine $f_{ac}(u, v, w)$ as

$$
\begin{aligned}
&f_{ac}(u, v, w)\\
&\quad = [\theta_2\theta_3\theta_4](0)\,\theta_1(2A_0)\,\frac{G_b}{G_{b-1}}\,\frac{\theta_1(B_{-b+c+1})\,\theta_4(B_{-b-c+1})}{\theta_2(A_{a-b})\,\theta_3(A_{-a-b})}\\
&\qquad \times \frac{W(a, b-1 \mid A_0)\,\bar{W}(b-1, c \mid B_0)}{W(a, b \mid A_{-1})\,\bar{W}(b, c \mid B_{-1})}\\
&\quad = C(u, v, w)\,\theta_1(2A_0)\,\frac{[\theta_1\theta_4](B_1)}{[\theta_2\theta_3](A_0)} = C(u, v, w)\,f(u, v, w)
\end{aligned}
$$

where $C(u, v, w)$ and $f(u, v, w)$ were given in (3.3) and (3.4), respectively. The last equality is due to (B.4). In the same way, we obtain

$$
g_{ac}(u, v, w) = -C(u, v, w)\,f(\lambda - v, \lambda - u, -w)
$$

Equations (3.24) and (3.25) become

$$
\begin{aligned}
&\phi^{(1)}(a, b, c \mid u, v, w + 1/4)\\
&\quad = C(u, v, w)\,f(u, v, w)\,W(a, b \mid A_{-1})\,\bar{W}(b, c \mid B_{-1}) \qquad (3.26)\\
&\phi^{(2)}(a, b, c \mid u, v, w + 1/4)\\
&\quad = -C(u, v, w)\,f(\lambda - v, \lambda - u, -w)\,W(a, b \mid A_1)\,\bar{W}(b, c \mid B_1) \qquad (3.27)
\end{aligned}
$$

Substituting (3.21), (3.26), and (3.27) into the definition (3.7) of $\psi^{(\nu)}(u, v, w)$ and using the crossing symmetry (2.13), we obtain (3.9)–(3.11). We have the functional relation (3.2) as a result.

4. BETHE ANSATZ EQUATIONS

In this section, we give commutation relations among $\mathscr{R}$, $\mathscr{L}(u)$, and $\Phi(v)$, and reduce the functional relation (3.2) to the functional equation among their eigenvalues. After discussing some properties about the zeros and poles of the eigenvalues of $\Phi(u)$, we derive the Bethe ansatz equations (1.7)–(1.9) for the broken $\mathbf{Z}_N$-symmetric model.

First we have

$$
\mathscr{L}(u, v, w')\,\Phi(u, v, w) = \Phi(u, v, w)\,\mathscr{L}(v, u, w') \qquad (4.1)
$$

We give a proof in Appendix C. In the case of the homogeneous systems, i.e., $u=v$ and $w=w'$, Eq. (4.1) means the commutativity of two transfer matrices $\mathscr{L}(u)=\mathscr{L}(u,u,w)$ and $\Phi(u)=\Phi(u,u,w)$,

$$[\mathscr{L}(u),\Phi(v)]=0 \tag{4.2}$$

The star–triangle relation (2.16) gives

$$[\Phi(u),\Phi(v)]=0 \tag{4.3}$$

and the $LLR=RLL$ relation (1.10) guarantees

$$[\mathscr{L}(u),\mathscr{L}(v)]=0 \tag{4.4}$$

The relations (4.2)–(4.4) make it possible to diagonalize $\mathscr{L}(u)$ and $\Phi(v)$ simultaneously by eigenvectors independent of the spectral parameters u and v. Fixing one of the eigenvectors and denoting the corresponding eigenvalues of $\mathscr{L}(u)$ and $\Phi(v)$ by $l(u)$ and $\varphi(v)$, respectively, we can rewrite the functional relation (3.2) as

$$\begin{aligned} &l(\lambda-u-1/4)\,\varphi(u) \\ &\quad=C(u)^M(f(u)^M\,\varphi(u-\eta)+(-1)^M f(\lambda-u)^M\,\varphi(u+\eta)) \end{aligned} \tag{4.5}$$

where from (3.4) and (3.3), $f(u)$ and $C(u)$ are

$$\begin{aligned} f(u)&=\theta_1(2u)\,\frac{[\theta_1\theta_4](u+\eta)}{[\theta_2\theta_3](u)} \\ C(u)&=[\theta_2\theta_3\theta_4](0)[T_2^{(+)}T_3^{(+)}T_2^{(-)}T_3^{(-)}](n\mid u) \end{aligned} \tag{4.6}$$

The next step to derive to derive the Bethe ansatz equations is to examine the quasiperiodicity property of $\varphi(u)$. We have the relations

$$\Phi(u+1)=\Phi(u) \tag{4.7}$$

$$\mathscr{R}\Phi(u)=\Phi(u)\mathscr{R}=\Phi\left(u+\frac{\tau}{2}\right) \tag{4.8}$$

We have defined $\mathscr{R}$ in (1.4). The periodicity (4.7) is obvious from the definition (1.2) of Φ and the periodicity (B.5) of W and $\bar{W}$. The other periodicity (4.8) follows from (B.6). We also have the commutativity

$$[\mathscr{R},\mathscr{L}(u)]=0 \tag{4.9}$$

from the $\mathbf{Z}_2$-symmetry of K's, (2.5), and the definitions (1.4) and (3.2).

Diagonalizing $\mathscr{R}$ and $\Phi(u)$ simultaneously with $\mathscr{L}(v)$, Eqs. (4.7), (4.8), and (1.4) give

$$\varphi(u+1)=\varphi(u), \qquad \varphi\left(u+\frac{\tau}{2}\right)=r\varphi(u), \qquad r=\pm 1 \tag{4.10}$$

where r is an eigenvalue of $\mathscr{R}$. The poles of $\varphi(u)$ are coming from only those of the matrix elements of $\Phi(u)$. We define

$$\begin{aligned} p(u) &= \left(\frac{\eta(\tau)}{\eta(2\tau)^2}\right)^n \prod_{j=1}^{n} [\theta_2\theta_3](u-(2j-1)\eta \mid \tau) \\ &= \prod_{j=1}^{n} \theta_2(u-(2j-1)\eta \mid \tau/2) \end{aligned} \tag{4.11}$$

which contains all possible poles of $W(a, b \mid u)$. The same is true for $p(\lambda-u)$ for $\bar{W}$ by the crossing symmetry. Hence the set of zeros of $(p(u)\,p(\lambda-u))^M$ contains all poles of $\varphi(u)$. By the Lemma in Appendix A and the double periodicity (4.10) of $\varphi(u)$, we can write $\varphi(u)$ as

$$\varphi(u)=(\text{const})\,\frac{\prod_{j=1}^{2nM}\theta_1(u-u_j \mid \tau/2)}{(p(u)\,p(\lambda-u))^M}$$

$$\sum_{j=1}^{2nM} u_j \equiv nM\lambda+\frac{1-r}{4} \mod \left(\mathbf{Z}\oplus\frac{\tau}{2}\mathbf{Z}\right)$$

The initial condition $\Phi(0)=Id$, (B.3) determines the normalization of const,

$$\varphi(u)=\left(\frac{p(0)\,p(\lambda)}{p(u)\,p(\lambda-u)}\right)^M \prod_{j=1}^{2nM} \frac{\theta_1(u-u_j \mid \tau/2)}{\theta_1(u_j \mid \tau/2)} \tag{4.12}$$

Assuming $C(u_j)\neq 0$ in (4.6) and substituting u_k ($k=1,\ldots, 2nM$) into (4.5), we have

$$f(u_k)^M\,\varphi(u_k-\eta)+(-1)^M f(\lambda-u_k)^M\,\varphi(u_k+\eta)=0 \qquad \text{for} \quad k=1,\ldots, 2nM \tag{4.13}$$

We further assume that u_k ($k=1,\ldots, 2nM$) are neither zeros nor poles of $f(u)$, $f(\lambda-u)$, and $\varphi(u\pm\eta)$. Then (4.13) becomes

$$\begin{aligned} &\left(\frac{f(u_k)}{f(\lambda-u_k)}\,\frac{p(u_k+\eta)\,p(\lambda-u_k-\eta)}{p(u_k-\eta)\,p(\lambda-u_k+\eta)}\right)^M \\ &\quad =(-1)^{M+1}\prod_{j=1}^{2nM}\frac{\theta_1(u_k-u_j+\eta \mid \tau/2)}{\theta_1(u_k-u_j-\eta \mid \tau/2)} \qquad \text{for} \quad k=1,\ldots, 2nM \end{aligned} \tag{4.14}$$

We can write $f(u)$ in (4.6) by (A.2) as

$$f(u)=\frac{\eta(2\tau)}{\eta(\tau)^2}\,\theta_1(u\mid\tau/2)\,\theta_1(u+\eta\mid\tau/2)$$

Then the left-hand side of (4.14) reduces to

$$\left(\frac{\theta_1(u_k\mid\tau/2)}{\theta_1(u_k-\lambda\mid\tau/2)}\right)^{2M}$$

After shifting u_k by $\lambda/2$, i.e., putting

$$v_k=u_k-\frac{\lambda}{2}\qquad\text{for}\quad k=1,\ldots,2nM$$

we obtain the Bethe ansatz equations (BAE) for the broken $\mathbf{Z}_N$-symmetric model,

$$\left(\frac{\theta_1(v_k+\lambda/2\mid\tau/2)}{\theta_1(v_k-\lambda/2\mid\tau/2)}\right)^{2M}=(-1)^{M+1}\prod_{j=1}^{2nM}\frac{\theta_1(v_k-v_j+\eta\mid\tau/2)}{\theta_1(v_k-v_j-\eta\mid\tau/2)}$$
$$\text{for}\quad k=1,\ldots,2nM \tag{4.15}$$

$$\sum_{j=1}^{2nM}v_j\equiv\frac{1-r}{4}\bmod\left(\mathbf{Z}\otimes\frac{\tau}{2}\mathbf{Z}\right) \tag{4.16}$$

5. DENSITY FUNCTION AND FREE ENERGY

In this section, we will calculate the free energy of the broken $\mathbf{Z}_N$-symmetric model from the Bethe ansatz equations under three assumptions concerning the ground state. One is the String Hypothesis below, and the others are about the distribution of string centers $v_\alpha=\sqrt{-1}\,w_\alpha$ and the corresponding quantum numbers I_α. We restrict the spectral parameter u to the region $[0, 1/2N]$, in which all the Boltzmann weights are real and positive, and τ to a pure imaginary number, $\tau=\sqrt{-1}\,\kappa$, with κ real and positive.

By a string of length l and parity ν ($=0$ or 1) with its center v_α, we mean the following set:

$$\left\{v_{\alpha,j}\;\middle|\;\begin{array}{l}v_{\alpha,j}\equiv v_\alpha+(2j-l-1)\dfrac{\eta}{2}+\dfrac{\nu}{2}\quad\bmod\left(\mathbf{Z}\oplus\dfrac{\tau}{2}\mathbf{Z}\right)\\[2mm]\text{for}\quad j=1,2,\ldots,l,\quad\text{and}\quad v_\alpha\ \text{pure imaginary}\end{array}\right\} \tag{5.1}$$

We suppose that the following hypothesis holds in the infinite lattice limit.[10,9,28]

String Hypothesis for the ground state. The solution of the BAE (4.15), $\{v_j, j=1,..., 2nM\}$, corresponding to the ground state consists of strings of length $N-1$ and parity $[1-(-1)^{n+1}]/2$.

More precisely, for finite systems the solutions of the BAE may have deviations from strings. The hypothesis asserts that these deviations vanish in the infinite lattice limit. In the course of the following calculation we deal with the solutions of the BAEs as if they were genuine strings, since we are interested in thermodynamic quantities. Because all the matrix elements of Φ are real and positive, the Perron–Frobenius theorem[29] shows that the ground state belongs to the sector of zero quasimomentum.[10] The hypothesis implies that the ground state also belongs to the sector $r=1$, and that the corresponding solutions are made up of M strings of length $2n$. We denote them by

$$v_{\alpha,j} \equiv \sqrt{-1}\, w_\alpha + (N-2j)\frac{\eta}{2} + \frac{1-(-1)^{n+1}}{2} \quad \mathrm{mod}\left(\mathbf{Z} \oplus \frac{\tau}{2}\mathbf{Z}\right)$$
$$\text{for} \quad \alpha = 1,..., M \quad \text{and} \quad j=1,..., 2n$$

where w_α are all real and taken as

$$-\frac{K}{4} \leqslant w_1 \leqslant w_2 \leqslant \cdots \leqslant w_M < \frac{K}{4}$$

Then the BAE for the ground state becomes

$$\left(\frac{\theta_1(v_{\beta,k}+\lambda/2 \mid \tau/2)}{\theta_1(v_{\beta,k}-\lambda/2 \mid \tau/2)}\right)^{2M} = (-1)^{M+1} \prod_{\alpha=1}^{M} \prod_{j=1}^{2n} \frac{\theta_1(v_{\beta,k}-v_{\alpha,j}+\eta \mid \tau/2)}{\theta_1(v_{\beta,k}-v_{\alpha,j}-\eta \mid \tau/2)}$$
$$\text{for} \quad \beta=1,..., M, \quad k=1,..., 2n \tag{5.2}$$

Multiplying (5.2) over $k=1,..., 2n$, we have

$$\left\{\prod_{k=1}^{2n} \chi\left(w_\beta, \frac{n}{2N}\left(n-2k+\frac{1}{2n}\right) + \frac{1-(-1)^{n+1}}{4}\right)\right\}^{2M}$$
$$\times \left(\prod_{\alpha=1}^{M} \left\{\chi(w_\beta - w_\alpha, 0)\left[\prod_{j=1}^{2n-1} \chi\left(w_\beta - w_\alpha, \frac{n}{N}j\right)\right]^2 \chi\left(w_\beta - w_\alpha, \frac{2n^2}{N}\right)\right\}\right)^{-1} = 1 \tag{5.3}$$

where $\chi(w, a)$ is

$$\chi(w, a) = \frac{\theta_1(a-\sqrt{-1}\, w \mid \tau/2)}{\theta_1(a+\sqrt{-1}\, w \mid \tau/2)}$$

Taking the logarithm of (5.3) and dividing it by $2\sqrt{-1}\,\pi M$, we have

$$\mathscr{T}(w_\beta)=\frac{I_\beta}{M} \qquad \text{for} \quad \beta=1,\dots,M \tag{5.4}$$

where the quantum numbers I_β are integers and

$$\mathscr{T}(w)=\mathscr{T}_1(w)-\frac{1}{2M}\sum_{\alpha=1}^{M}\mathscr{T}_2(w-w_\alpha)$$

$$\mathscr{T}_1(w)=\sum_{k=1}^{2n} t\left(w,\frac{n}{2N}\left(n-2k+\frac{1}{2n}\right)+\frac{1-(-1)^{n+1}}{4}\right)$$

$$\mathscr{T}_2(w)=t(w,0)+2\sum_{j=1}^{2n} t\left(w,\frac{n}{N}j\right)-t\left(w,\frac{2n^2}{N}\right)$$

$$t(w,a)=\frac{1}{\sqrt{-1}\,\pi}\log\chi(w,a)$$

Albertini *et al.*[9] numerically investigated the three-state Fateev–Zamolodchikov model. Their results indicate that the String Hypothesis holds. We assume that the centers w of the strings are distributed densely on the interval $[-\kappa/4,\kappa/4]$ in the limit of M large, and that the quantum numbers I_β satisfy

$$I_{\beta+1}=I_\beta+1 \qquad \text{for} \quad \beta=1,2,\dots,2nM \tag{5.5}$$

These are the second and the third assumptions we make. The results by Albertini *et al.* also support them. We furthermore conjecture that

$$w_\alpha=-w_{M-\alpha+1} \tag{5.6}$$

holds exactly for the ground state even in the finite lattice. This conjecture is conssistent with their results. If (5.6) is true, we can show that

$$\mathscr{T}(\kappa/4)-\mathscr{T}(-\kappa/4)=1 \tag{5.7}$$

and this implies that $2M$ integers I_β must fill the interval $[-M,M)$ without jumps if all I_β are different. This also supports our assumption. But we do not use the conjecture (5.6) in this paper.

We now proceed to the calculation. We define the density function for w's by

$$\rho(w_\beta)=\lim_{M\to\infty}\frac{1}{M(w_{\beta+1}-w_\beta)} \tag{5.8}$$

which is positive and for any integrable function $f(x)$,

$$\lim_{M\to\infty}\frac{1}{M}\sum_{\alpha=1}^{M} f(w_\alpha)=\int_{-\kappa/4}^{\kappa/4} f(w)\,\rho(w)\,dw \tag{5.9}$$

holds. Considering the difference of (5.4) for $I_{\beta+1}$ and I_β

$$\frac{1}{M(w_{\beta+1}-w_\beta)}=\frac{I_{\beta+1}-I_\beta}{M(w_{\beta+1}-w_\beta)}=\frac{\mathscr{T}(w_{\beta+1})-\mathscr{T}(w_\beta)}{w_{\beta+1}-w_\beta}$$

and letting $M\to\infty$, we obtain

$$\rho(w)=\frac{d\mathscr{T}(w)}{dw}=\frac{d\mathscr{T}_1(w)}{dw}-\frac{1}{2}\int_{-\kappa/4}^{\kappa/4}\frac{d\mathscr{T}_2(w-\bar{w})}{dw}\rho(\bar{w})\,d\bar{w} \tag{5.10}$$

By (A.1) and (A.4), we can expand $t(w,a)$ as

$$\begin{aligned} t(w,a)&=\frac{8\pi}{\kappa}\{a\}\,w+\frac{1}{\sqrt{-1}}\log\frac{\theta_1((2\sqrt{-1}/\kappa)\{a\}+(2/\kappa)w\mid -2/\tau)}{\theta_1((2\sqrt{-1}/\kappa)\{a\}-(2/\kappa)w\mid -2/\tau)}\\ &=\frac{4\pi}{\kappa}(2\{a\}-1)+\sum_{k=1}^{\infty}\frac{\sin((4\pi k/\kappa)w)\sinh((4\pi k/\kappa)(\{a\}-1/2))}{k\sinh(2\pi k/\kappa)} \end{aligned}$$

We denote the fractional part of x by $\{x\}=x-[x]$, $[x]$ being the Gauss symbol. When we write the Fourier expansions of $d\mathscr{T}_1(w)/dw$, $d\mathscr{T}_2(w)/dw$, and $\rho(w)$ as

$$\frac{d\mathscr{T}_j(w)}{dw}=\sum_{k=-\infty}^{\infty}A_{jk}\exp\left(\frac{4\sqrt{-1}\,\pi k}{\kappa}w\right)\qquad \text{for}\quad j=1,2$$

$$\rho(w)=\sum_{k=-\infty}^{\infty}\rho_k\exp\left(\frac{4\sqrt{-1}\,\pi k}{\kappa}w\right)$$

the integral equation (5.10) gives

$$\rho_k=\frac{A_{1,k}}{1+(\kappa/4)\,A_{2,k}}$$

The coefficients A_{jk} are

$$A_{1,k}=\begin{cases}\dfrac{4n}{N\kappa} & k=0\\[2mm] \dfrac{4\sinh((\pi k/N\kappa)(N-1))\cosh((\pi k/N\kappa)(N+1))}{\kappa\sinh(2\pi k/\kappa)\cosh(2\pi k/\kappa)} & k\neq 0\end{cases}$$

$$1+\frac{\kappa}{4}A_{2,k}=\begin{cases}\dfrac{2n}{N} & k=0\\[2ex] \dfrac{2\sinh((\pi k/N\kappa)(N-1))\cosh((\pi k/N\kappa)(N+1))}{\sinh(2\pi k/\kappa)} & k\neq 0\end{cases}$$

We obtain the density function for strings,

$$\rho(w)=\frac{2}{\kappa}\sum_{k=-\infty}^{\infty}\frac{\exp((4\sqrt{-1}\,\pi k/\kappa)w)}{\cosh(\pi k/N\kappa)}$$

Using (A.1) and (A.5), we can rewrite it as

$$\begin{aligned}\rho(w)&=\frac{2}{\kappa}[\theta_3\theta_4]\left(0\,\middle|\,-\frac{1}{N\tau}\right)\frac{\theta_3(2\omega/\kappa\mid -1/N\tau)}{\theta_4(2w/\kappa\mid -1/N\tau)}\\ &=2N[\theta_2\theta_3](0\mid N\tau)\frac{\theta_3(2\sqrt{-1}\,Nw\mid N\tau)}{\theta_2(2\sqrt{-1}\,Nw\mid N\tau)}\end{aligned}\tag{5.11}$$

The free energy per site $F(u)$ of the model is defined by

$$F(u)=\lim_{M\to\infty}\left(-\frac{1}{M}\log\varphi(u)\right)\tag{5.12}$$

where $\varphi(u)$ is the eigenvalue of the transfer matrix $\Phi(u)$ corresponding to the ground state. We can write $\varphi(u)$ as

$$\varphi(u)=\left(\frac{p(0)\,p(\lambda)}{p(u)\,p(\lambda-u)}\right)^M\prod_{j=1}^{2n}\prod_{\alpha=1}^{M}D_j(u,w_\alpha)$$

$$D_j(u,w)=\frac{\theta_1(\sqrt{-1}\,w+\beta_j-u\mid\tau/2)}{\theta_1(\sqrt{-1}\,w+\beta_j\mid\tau/2)}$$

$$\beta_j=\{\gamma_j\},\qquad \gamma_j=\frac{n}{2N}\left(N-2j+\frac{1}{2n}\right)+\frac{1-(-1)^{n+1}}{4}$$

by (4.12), (5.6), and the String Hypothesis. Then the free energy per site is

$$\begin{aligned}F(u)=&-\log\left(\frac{p(0)\,p(\lambda)}{p(u)\,p(\lambda-u)}\right)\\ &-\sum_{j=1}^{2n}\lim_{M\to\infty}\frac{1}{M}\sum_{\alpha=1}^{M}\log D_j(u,w_\alpha)\end{aligned}$$

$$= -\log\left(\frac{p(0)\,p(\lambda)}{p(u)\,p(\lambda-u)}\right)$$
$$-\frac{1}{2}\sum_{j=1}^{2n}\int_{-\kappa/4}^{\kappa/4}(\log D_j(u,w))\,\rho(w)\,dw$$

Since $\rho(w)$ is an even function, it is enough to integrate the even part of $\log D_j(u,w)$. We hence have

$$F(u) = -\log\left(\frac{p(0)\,p(\lambda)}{p(u)\,p(\lambda-u)}\right)$$
$$-\frac{1}{2}\sum_{j=1}^{2n}\log D_j^{(e)}\left(u,\frac{\kappa}{4}\right)$$
$$+\frac{1}{2}\sum_{j=1}^{2n}\int_{-\kappa/4}^{\kappa/4}\frac{d\log D_j^{(e)}(u,w)}{dw}\,\rho^{(-1)}(w)\,dw$$

where $D_j^{(e)}(u,w)$ and $\rho^{(-1)}(w)$ are

$$\rho^{(-1)}(w) = \int_{-\kappa/4}^{w}\rho(\bar{w})\,d\bar{w}$$
$$D_j^{(e)}(u,w) = (D_j(u,w)\,D_j(u,-w))^{1/2}$$

A little cumbersome calculation yields

$$-\log\left(\frac{p(0)\,p(\lambda)}{p(u)\,p(\lambda-u)}\right) = E_1(u)+E_2(u)$$
$$-\frac{1}{2}\sum_{j=1}^{2n}\log D_j^{(e)}\left(u,\frac{\kappa}{4}\right) = -E_1(u)+E_3(u)$$
$$\frac{1}{2}\sum_{j=1}^{2n}\int_{-\kappa/4}^{\kappa/4}\frac{d\log D_j^{(e)}(u,w)}{dw}\,\rho^{(-1)}(w)\,dw = -E_3(u)+E_4(u)$$

where

$$E_1(u) = \frac{2\pi n}{\kappa}\,u\left(\frac{1}{N}-2u\right)$$
$$E_2(u) = 4\sum_{l=1}^{\infty}\left(\left\{\sinh\left(\frac{2\pi l}{\kappa}u\right)\sinh\left[\frac{2\pi l}{\kappa}\left(\frac{1}{2N}-u\right)\right]\right.\right.$$
$$\left.\times\cosh\left(\frac{\pi l}{\kappa}\right)\sinh\left(\frac{2\pi l}{N\kappa}n\right)\right\}$$
$$\left.\times\left[l\sinh\left(\frac{2\pi l}{\kappa}\right)\sinh\left(\frac{2\pi l}{N\kappa}\right)\right]^{-1}\right)$$

$$E_3(u) = -4 \sum_{l=1}^{\infty} (-1)^l \left(\left\{ \sinh\left(\frac{2\pi l}{\kappa} u\right) \sinh\left[\frac{2\pi l}{\kappa}\left(\frac{1}{2N} - u\right)\right] \right.\right.$$
$$\left. \times \cosh\left[\frac{2\pi l}{N\kappa}(n+1)\right] \sinh\left(\frac{2\pi l}{N\kappa} n\right)\right\}$$
$$\left. \times \left[l \sinh\left(\frac{2\pi l}{\kappa}\right) \sinh\left(\frac{2\pi l}{N\kappa}\right)\right]^{-1}\right)$$
$$E_4(u) = 4 \sum_{l=1}^{\infty} \left(\left\{ \sinh\left(\frac{2\pi l}{\kappa} u\right) \sinh\left[\frac{2\pi l}{\kappa}\left(u - \frac{1}{2N}\right)\right] \right.\right.$$
$$\left. \times \cosh\left[\frac{2\pi l}{N\kappa}(n+1)\right] \sinh\left(\frac{2\pi l}{N\kappa} n\right)\right\}$$
$$\left. \times \left[l \sinh\left(\frac{2\pi l}{\kappa}\right) \sinh\left(\frac{2\pi l}{N\kappa}\right) \cosh\left(\frac{\pi l}{N\kappa}\right)\right]^{-1}\right)$$

Now $F(u)$ has the final expression

$$F(u) = E_2(u) + E_4(u)$$
$$= -\sum_{l=1}^{\infty} \left(\left\{ \sinh\left(\frac{2\pi l}{\kappa} u\right) \sinh\left[\frac{2\pi l}{\kappa}\left(\frac{1}{2N} - u\right)\right] \sinh\left(\frac{2\pi l}{N\kappa} n\right)\right\} \right.$$
$$\left. \times \left[l \cosh\left(\frac{\pi l}{\kappa}\right) \cosh^2\left(\frac{\pi l}{N\kappa}\right)\right]^{-1}\right) \tag{5.13}$$

It agrees with the result of Jimbo *et al.*[20] obtained by the use of the inversion trick. In the trigonometric limit $\kappa \to +\infty$, the free energy formula (5.13) reduces to the integral

$$\lim_{\kappa \to \infty} F(u) = -\int_0^{\infty} dx \, \frac{\sinh(N\pi x u) \sinh[N\pi x(1/2N - u)] \sinh(n\pi x)}{x \cosh(N\pi x/2) \cosh^2(\pi x/2)}$$

which also agrees with the results of Fateev and Zamolodchikov[1] and Albertini.[10] The former was obtained by the inversion trick, and the latter by the Bethe ansatz method.

6. DISCUSSION

The first main goal of this paper is the functional relation (3.2). We obtain it was a functional relation for $\mathscr{L}(u)$. The diagonal-to-diagonal transfer matrix $\Phi(u)$ of the broken $\mathbf{Z}_N$-symmetric model appears naturally in this relation. We obtain the Boltzmann weights W and $\bar{W}$ of the broken

$\mathbf{Z}_N$-symmetric model in an algebraic way different from that of ref. 12. We obtain W and $\bar{W}$ as the solutions to the relation (3.12). In ref. 12 they are the solutions to the relations (2.9) and (2.10).

The Bethe ansatz equations (4.15) are the second goal of this paper. The commutativity (4.1) between $\mathscr{L}(u)$ and $\Phi(v)$ is essential to get the Bethe ansatz equations (4.15) from the functional relation (3.2). It is notable that the unitarity relations (2.6) and (2.7) guarantee this commutativity. This contrasts with the usual situation where the commutativity of the transfer matrices is derived from the STR or the $LLR = RLL$ type relations.

The Fateev–Zamolodchikov model is the trigonometric limit of the broken $\mathbf{Z}_N$-symmetric model. It has the $\mathbf{Z}_N$-symmetry besides the $\mathbf{Z}_2$-symmetry. Hence the $\mathbf{Z}_N$-charge $q \in \{-n, -n+1, ..., n-1, n\}$ is a good quantum number, where $\exp(2\sqrt{-1}\,\pi(q/N)$ is an eigenvalue of the $\mathbf{Z}_N$-charge operator $\mathscr{Q} \in \mathrm{End}((\mathbf{C}^N)^{\otimes M})$,

$$\mathscr{Q} = \overbrace{Q \otimes Q \otimes \cdots \otimes Q}^{M \text{ times}}$$

$$Qv_j^{(N)} = \exp\left(2\sqrt{-1}\,\pi\frac{j}{N}\right)v_j^{(N)} \qquad \text{for} \quad j = 0, 1, ..., N-1 \tag{6.1}$$

Albertini[10] obtained the Bethe ansatz equations and the formula for the eigenvalue $\varphi_{\mathrm{FZ}}(u)$ of the diagonal-to-diagonal transfer matrix for the Fateev–Zamolodchikov model. They are

$$\left(\frac{s(v_k + \lambda/2)}{s(v_k - \lambda/2)}\right)^{2M} = (-1)^{M+1} \prod_{j=1}^{2nM-2|q|} \frac{s(v_k - v_j + \eta)}{s(v_k - v_j - \eta)}$$

$$\text{for} \quad k = 1, ..., 2nM \tag{6.2}$$

$$\varphi_{\mathrm{FZ}}(u) = \left(\frac{p_\infty(0)\, p_\infty(\lambda)}{p_\infty(u)\, p_\infty(\lambda - u)}\right)^M \prod_{j=1}^{2nM-2|q|} \frac{s(u - u_j)}{s(u_j)}$$

$$p_\infty(u) = \lim_{\kappa \to \infty} p(u) \tag{6.3}$$

where $s(u) = \sin(\pi u)$ and $p(u)$ is given in (4.11). The notations are slightly changed from ref. 10 to allow comparison to our case. The main difference is the number of factors on the right-hand sides of the BAE and in the expressions of the eigenvalues $\varphi(u)$ and $\varphi_{\mathrm{FZ}}(u)$. It is always $2nM$ in the broken $\mathbf{Z}_N$-symmetric model, and in the Fateev–Zamolodchikov model it is $2nM - 2|q|$, which depends on the sector of the $\mathbf{Z}_N$-charge operator $\mathscr{Q}$. This difference originates in the fact that the $\mathbf{Z}_N$-symmetry holds only in the Fateev–Zamolodchikov model and that it breaks away from the criticality. The BAE (4.15) for the broekn $\mathbf{Z}_N$-symmetric model should coincide with those for the Fateev–Zamolodchikov model in the trigonometric limit $\kappa \to \infty$. We conjecture that the situation is the following. In the solution $\{v_1, \ldots, v_{2nM}\}$ to the BAE (4.15) for the broken $\mathbf{Z}_N$-symmetric model, some of them diverge to $\pm\sqrt{-1}\,\infty$ all in the same order in κ when the trigonometric limit is taken. Half of them diverge to $\sqrt{-1}\,\infty$, and the other half to $-\sqrt{-1}\,\infty$. There is always an even number of them, between 0 and $2n$. Let $2q$ be this number. Then q determines the sector of the $\mathbf{Z}_N$-charge operator in which this eigenvalue falls. In this situation, the BAE (4.15) and the eigenvalue $\varphi(u)$ in (4.12) surely become the BAE (6.2) and $\varphi_{\mathrm{FZ}}(u)$ in (6.3), respectively, in the trigonometric limit.

The free energy (5.13) agrees with the result of ref. 20 by the inversion trick. The String Hypothesis for the ground state in Section 5 is consistent with their result. In our formulation, it is manifest that the free energy $F(u)$ is doubly periodic in u,

$$F(u+1) = F(u+\tau/2) = F(u)$$

from (4.10), (5.12), and the fact that the ground state belongs to the sector of $r=1$. This result of for $F(u)$ also gives the ground-state energy of the one-dimensional spin chain Hamiltonian $\mathscr{H}$ corresponding to the broken $\mathbf{Z}_N$-symmetric model,

$$\log \Phi(u) = Id + u\mathscr{H} + O(u^2)$$

The Hamiltonian $\mathscr{H}$ itself is modular invariant, $\mathscr{H}(\tau) = \mathscr{H}(-1/\tau)$. We will report on these matters elsewhere.

APPENDIX A. THETA FUNCTION

We summarize the necessary facts about theta functions in this appendix. See refs. 30 and 31 for proofs. We define $\theta_1(u \mid \tau)$ by

$$\theta_1(u \mid \tau) = 2q^{1/4} \sin(\pi u) \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos(2\pi u) + q^{4n})(1 - q^{2n})$$

where $q = \exp(\sqrt{-1}\,\pi\tau)$. It is an odd function in u and has the quasiperiodicity

$$\theta_1(u+1 \mid \tau) = \theta_1(u \mid \tau)$$
$$\theta_1(u+\tau \mid \tau) = -q^{-1} \exp(2\sqrt{-1}\,\pi u)\, \theta_1(u \mid \tau)$$

and satisfies

$$\theta_1(u \mid \tau) = \sqrt{-1}\left(\frac{\sqrt{-1}}{\tau}\right)^{1/2} \exp\left(-\sqrt{-1}\,\pi \frac{u^2}{\tau}\right) \theta_1\left(\frac{u}{\tau}\,\middle|\, -\frac{1}{\tau}\right) \tag{A.1}$$

The other theta functions θ_2, θ_3, and θ_4 are defined by

$$\begin{aligned}
\theta_2(u \mid \tau) &= \theta_1(u + 1/2 \mid \tau) \\
\theta_3(u \mid \tau) &= -q^{1/4} \exp(\sqrt{-1}\,\pi u)\, \theta_1(u + 1/2 + \tau/2 \mid \tau) \\
\theta_4(u \mid \tau) &= -\sqrt{-1}\, q^{1/2} \exp(\sqrt{-1}\,\pi u)\, \theta_1(u + \tau/2 \mid \tau)
\end{aligned}$$

We abbreviate a product of theta functions of the same argument to, for example,

$$\begin{aligned}
[\theta_1 \theta_2](u \mid \tau) &= \theta_1(u \mid \tau)\, \theta_2(u \mid \tau) \\
[\theta_2 \theta_3 \theta_4](0 \mid \tau) &= \theta_2(0 \mid \tau)\, \theta_3(0 \mid \tau)\, \theta_4(0 \mid \tau)
\end{aligned}$$

In this notation,

$$\begin{aligned}
[\theta_1 \theta_4](u \mid \tau) &= \frac{\eta(2\tau)^2}{\eta(\tau)}\, \theta_1(u \mid \tau/2), \qquad & [\theta_2 \theta_3](u \mid \tau) &= \frac{\eta(2\tau)^2}{\eta(\tau)}\, \theta_2(u \mid \tau/2) \\
[\theta_1 \theta_2](u \mid \tau) &= \frac{\eta(2\tau)^2}{\eta(4\tau)}\, \theta_1(2u \mid 2\tau), \qquad & [\theta_3 \theta_4](u \mid \tau) &= \frac{\eta(2\tau)^2}{\eta(4\tau)}\, \theta_4(2u \mid 2\tau)
\end{aligned} \tag{A.2}$$

hold, where

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind eta function. The necessary addition formulas are

$$\begin{aligned}
&[\theta_2 \theta_3](u \mid \tau)[\theta_2 \theta_3](v \mid \tau) \pm [\theta_1 \theta_4](v \mid \tau)[\theta_1 \theta_4](u \mid \tau) \\
&\qquad = [\theta_2 \theta_3](0 \mid \tau)\, \theta_2(u \mp v \mid \tau)\, \theta_3(u \pm v \mid \tau) \\
&[\theta_1 \theta_4](u \mid \tau)[\theta_2 \theta_3](v \mid \tau) \pm [\theta_2 \theta_3](u \mid \tau)[\theta_1 \theta_4](v \mid \tau) \\
&\qquad = [\theta_2 \theta_3](0 \mid \tau)\, \theta_1(u \pm v \mid \tau)\, \theta_4(u \mp v \mid \tau)
\end{aligned} \tag{A.3}$$

In Section 5, we use the Fourier expansions

$$\frac{1}{\sqrt{-1}} \log \frac{\theta_1(w+v \mid \tau)}{\theta_1(w-v \mid \tau)} = -2\pi\{v\} + 2 \sum_{k=1}^{\infty} \frac{\sin(2\pi kv)\sin(2\pi k(w-\tau/2))}{k \sin(\pi k\tau)} \tag{A.4}$$

$$\frac{\theta_3(u \mid \tau)}{\theta_4(u \mid \tau)} = \frac{1}{[\theta_3\theta_4](0 \mid \tau)} \sum_{k=-\infty}^{\infty} \frac{\exp(\sqrt{-1}\,\pi ku)}{\cos(\pi k\tau)} \tag{A.5}$$

which are valid for $0 < \mathrm{Im}(w) < \tau$ and v real. We are denoting the fractional part of x by $\{x\}$. The expansion (A.5) is essentially the same as that of the Jacobi elliptic function $\mathrm{dn}(n, k)$

$$\mathrm{dn}(u, k) = \frac{\pi}{2K} \sum_{l=-\infty}^{\infty} \frac{\exp(\sqrt{-1}\,\pi lu/K)}{\cos(\pi l\tau)}, \qquad K = \frac{\pi}{2}\theta_3(0 \mid \tau)^2$$

The next lemma is fundamental.

Lemma 1. Let $f(u)$ be a meromorphic function which is not identically zero and has the quasiperiodicity property

$$f(u+1) = \exp(-2\sqrt{-1}\,\pi B) f(u)$$
$$f(u+\tau) = \exp(-2\sqrt{-1}\,\pi(A_1 + A_2 u)) f(u)$$

Denoting the zeros and poles of $f(u)$ by $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$, respectively, then we have

$$n - m = A_2, \qquad \sum_{j=1}^{n} u_j - \sum_{j=1}^{m} v_j \equiv \tfrac{1}{2}A_2 - A_1 + B\tau \mod(\mathbf{Z} \oplus \tau\mathbf{Z})$$

and

$$f(u) = C \exp(\sqrt{-1}\,\pi(A_2 - 2B)u) \frac{\prod_{j=1}^{n} \theta_1(u - u_j \mid \tau)}{\prod_{j=1}^{m} \theta_1(u - v_j \mid \tau)}$$

with C independent of u.

APPENDIX B. BOLTZMANN WEIGHTS

In this apendix, we list some formulas for the Boltzmann weights. Solving the recursion relations (2.12) and (2.14) under the normalization of $W(0, 0 \mid u) = \bar{W}(0, 0 \mid u) = 1$, we have, for $a, b = 0, 1, \ldots, n$,

$$\begin{aligned}
W(2a, 2b \mid u) &= W(N-2a, N-2b \mid u) \\
&= T_2^{(+)}(|a-b| \mid u)\, T_3^{(+)}(a+b \mid u) \\
W(2a, N-2b \mid u) &= W(N-2a, 2b \mid u) \\
&= T_2^{(+)}(a+b \mid u)\, T_3^{(+)}(|a-b| \mid u) \\
\bar{W}(2a, 2b \mid u) &= \bar{W}(N-2a, N-2b \mid u) \\
&= G_{2a} G_{2b} T_2^{(-)}(|a-b| \mid u)\, T_3^{(-)}(a+b \mid u) \\
\bar{W}(2a, N-2b \mid u) &= \bar{W}(N-2a, 2b \mid u) \\
&= G_{2a} G_{2b} T_2^{(-)}(a+b \mid u)\, T_3^{(-)}(|a-b| \mid u)
\end{aligned}$$

Noting that $T_k^{(\sigma)}(a \mid u)$ in (2.15) satisfies

$$T_k^{(\sigma)}(0 \mid u) = T_k^{(\sigma)}(N \mid u) \tag{B.1}$$

$$T_k^{(\sigma)}(N-a \mid u) = T_k^{(\sigma)}(a \mid u) \tag{B.2}$$

we can extend the domain of the first argument of $T_k^{(\sigma)}$ to all integers by periodicity. With this convention we can rewrite the above expressions for the Boltzmann weights simply as

$$\begin{aligned}
W(2a, 2b \mid u) &= T_2^{(+)}(a-b \mid u)\, T_3^{(+)}(a+b \mid u) \\
\bar{W}(2a, 2b \mid u) &= G_{2a} G_{2b} T_2^{(-)}(a-b \mid u)\, T_3^{(-)}(a+b \mid u)
\end{aligned}$$

We have in particular at $u=0$ and λ,

$$\begin{aligned}
W(a, b \mid 0) &= G_a^{-1} G_b^{-1} \bar{W}(a, b \mid \lambda) = 1 \\
G_a G_b W(a, b \mid \lambda) &= \bar{W}(a, b \mid 0) = \delta_{ab}
\end{aligned} \tag{B.3}$$

When we write $u + a\eta$ as U_a, the next identity for $T_k^{(\sigma)}$

$$\frac{T_k^{(\sigma_1)}(n-a \mid u)}{T_k^{(\sigma_1)}(a \mid u+\sigma_2\eta)} = \left(\frac{\theta_k(U_{(1+\sigma_1)/2})}{\theta_k(U_{-2\sigma_2+(1+\sigma_1)/2})}\right)^{\sigma_1\sigma_2} T_k^{(\sigma_1)}(n \mid u)$$

gives

$$\begin{aligned}
\frac{W(a, b-1 \mid u)}{W(a, b \mid u+\sigma\eta)} &= (\theta_2(U_{\sigma(-a+b)})\, \theta_3(U_{\sigma(a+b)}))^{-\sigma} [T_2^{(+)} T_3^{(+)}](n \mid u) \\
\frac{\bar{W}(a, b-1 \mid u)}{\bar{W}(a, b \mid u+\sigma\eta)} &= \frac{G_{b-1}}{G_b} (\theta_1(U_{\sigma(-a+b)+1})\, \theta_4(U_{\sigma(a+b)+1}))^{\sigma} [T_2^{(-)} T_3^{(-)}](n \mid u)
\end{aligned} \tag{B.4}$$

$T_k^{(\sigma)}$ has the quasiperiodicity

$$T_k^{(\sigma)}(a \mid u+1) = \exp(4\sqrt{-1}\,\pi\sigma a^2\eta)\, T_k^{(\sigma)}(a \mid u+\tau) = T_k^{(\sigma)}(a \mid u)$$
$$T_k^{(\sigma)}(a \mid u+\tau/2) = \exp(-2\sqrt{-1}\,\pi\sigma a^2\eta)\, T_{k'}^{(\sigma)}(a \mid u), \qquad k+k' \equiv 0 \bmod 5$$

Hence we have

$$W(a,b \mid u+1) = W(a,b \mid u), \qquad \bar{W}(a,b \mid u+1) = \bar{W}(a,b \mid u) \tag{B.5}$$

and

$$W(a,b \mid u+\tau/2) = \exp(-4\sqrt{-1}\,\pi(a^2+b^2)\eta)\, W(a, N-b \mid u)$$
$$\bar{W}(a,b \mid u+\tau/2) = \exp(4\sqrt{-1}\,\pi(a^2+b^2)\eta)\, \bar{W}(a, N-b \mid u)$$

We note

$$\begin{aligned} &W(a,b \mid u)\, \bar{W}(b,c \mid u) \\ &\quad = \exp(-4\sqrt{-1}\,\pi(a^2-c^2)\eta)\, W(a, N-b \mid u+\tau/2)\, \bar{W}(N-b, c \mid u+\tau/2) \end{aligned} \tag{B.6}$$

APPENDIX C. COMMUTATION RELATION BETWEEN $\mathscr{L}$ AND Φ

In this appendix, we give a proof of (4.1). A graphical representation of this proof for the case of $M=2$ is illustrated in Fig. 11. We have

$$\begin{aligned} &(\Phi(u,v,w)\, \mathscr{L}(u,v,w'))_{a_0 \cdots a_{M-1}}^{c_0 \cdots c_{M-1}} \\ &\quad = \sum_{b_0 \cdots b_{M-1}} \Phi(u,v,w)_{b_0 \cdots b_{M-1}}^{c_0 \cdots c_{M-1}}\, \mathscr{L}(u,v,w')_{a_0 \cdots a_{M-1}}^{b_0 \cdots b_{M-1}} \\ &\quad = \sum_{\substack{b_0 \cdots b_{M-1} \\ i_0 \cdots i_{M-1}}} \left(\prod_{j=0}^{M-1} W(b_{j-1}, c_j \mid u-w)\, \bar{W}(c_j, b_j \mid v-w) \right. \\ &\qquad \left. \times K_{i_{j+1}\,a_j}^{\;\;\;\;\;b_j}(u-w')\, K^{i_j\,b_j}_{\;\;\;a_j}(v-w') \right) \end{aligned}$$

By the unitarity relation (2.7), inserting

$$1 = \sum_{i'} \delta_{i_0 i'} = \frac{\sum_{i'a'} G_{a'} K_{i'\,a'}^{\;\;c_0}(w-w'+\lambda)\, K^{i_0\,c_0}_{\;\;\;a'}(w-w')}{[\theta_2\theta_3\theta_4](0)\, G_{c_0}\theta_2(2w-2w')}$$

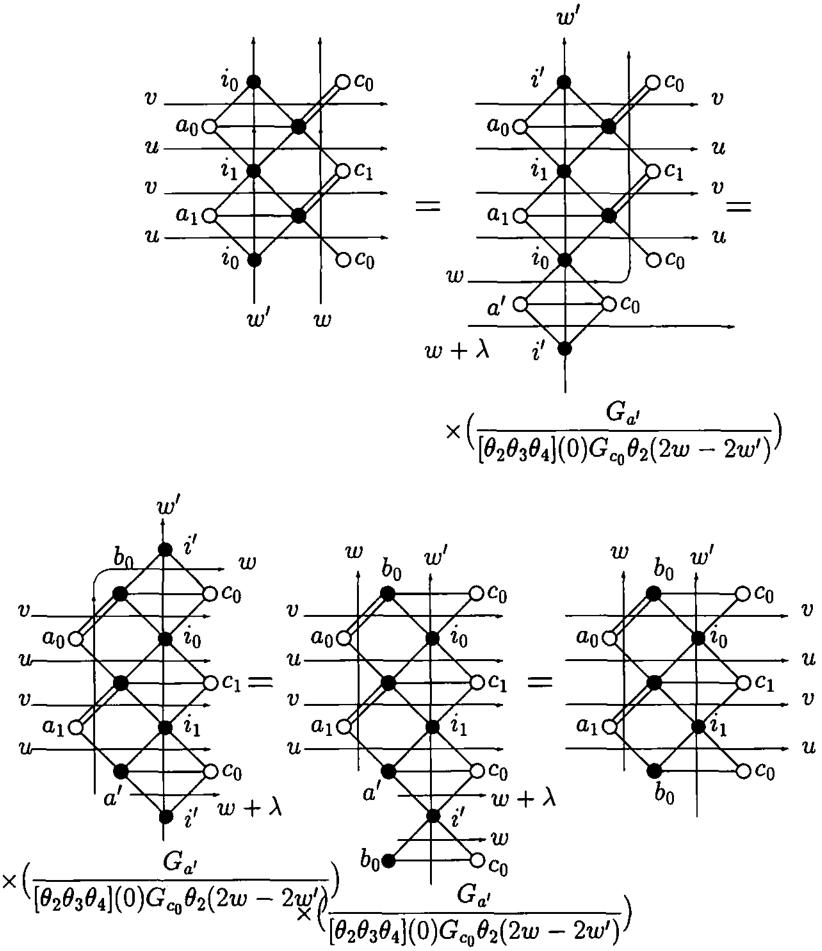


Fig. 11. Graphical representation of a proof of $\mathscr{L}(u, v, w')\,\Phi(u, v, w) = \Phi(u, v, w)\,\mathscr{L}(u, v, w')$.

we have

$$
\begin{aligned}
&= ([\theta_2\theta_3\theta_4](0)\; G_{c_0}\theta_2(2w-2w'))^{-1} \\
&\quad\times \sum_{\substack{b_0\cdots b_{M-1}\\ i_0\cdots i_{M-1}}} \cdot \sum_{i'a'} G_{a'} K_{i'\,a'}^{\,c_0}(w-w'+\lambda) \\
&\quad\times\left(\sum_{j=0}^{M-1} \bar{W}(c_j, b_j \mid v-w)\; W(b_{j-1}, c_j \mid u-w)\right)
\end{aligned}
$$

$$\times K^{i_0\,c_0}_{\;\;a'}(w-w')\,K_{i_0\,a_{M-1}}^{\;\;b_{M-1}}(u-w')$$
$$\times\left(\prod_{j=1}^{M-1} K^{i_j\,b_j}_{\;\;a_j}(v-w')\,K_{i_j\,a_{j-1}}^{\;\;b_{j-1}}(u-w')\right)K^{i'\,b_0}_{\;\;a_0}(v-w')$$

Successive use of (2.9) and (2.10) yields

$$=([\theta_2\theta_3\theta_4](0)\,G_{c_0}\theta_2(2w-2w'))^{-1}$$
$$\times\sum_{\substack{b_0\cdots b_{M-1}\\ i_0\cdots i_{M-1}}}\sum_{i'a'} G_{a'}K_{i'\,a'}^{\;\;c_0}(w-w'+\lambda)\,K^{i'\,c_0}_{\;\;b_0}(w-w')$$
$$\times\left(\sum_{j=0}^{M-2} K_{i_j\,b_j}^{\;\;c_j}(v-w')\,K^{i_j\,c_{j+1}}_{\;\;b_{j+1}}(u-w')\right)$$
$$\times K_{i_{M-1}\,b_{M-1}}^{\;\;c_{M-1}}(v-w')\,K^{i_{M-1}\,c_0}_{\;\;a'}(u-w')$$
$$\times\left(\prod_{j=0}^{M-2}\bar{W}(b_j,a_j\mid v-w)\,W(a_j,b_{j+1}\mid u-w)\right)$$
$$\times\bar{W}(b_{M-1},a_{M-1}\mid v-w)\,W(a_{M-1},a'\mid u-w)$$

By the unitarity relation (2.6), we have

$$\frac{\sum_{i'} G_{a_0}K_{i'\,a'}^{\;\;c_0}(w-w'+\lambda)\,K^{i'\,c_0}_{\;\;b_{M-1}}(w-w')}{[\theta_2\theta_3\theta_4](0)\,G_{c_0}\theta_2(2w-2w')}=\delta_{a'b_{M-1}}$$

then the above formula reduces

$$=\sum_{\substack{b_0\cdots b_{M-1}\\ i_0\cdots i_{M-1}}}\left(\sum_{j=0}^{M-1} K_{i_j\,b_j}^{\;\;c_j}(v-w')\,K^{i_j\,c_{j+1}}_{\;\;b_{j+1}}(u-w')\right.$$
$$\left.\times\bar{W}(b_j,a_j\mid v-w)\,W(a_j,b_{j+1}\mid u-w)\right)$$
$$=\sum_{b_0\cdots b_{M-1}}\mathscr{L}(u,v,w')^{c_0\cdots c_{M-1}}_{b_0\cdots b_{M-1}}\,\Phi(u,v,w)^{b_0\cdots b_{M-1}}_{a_0\cdots a_{M-1}}$$
$$=(\mathscr{L}(u,v,w')\,\Phi(u,v,w))^{c_0\cdots c_{M-1}}_{a_0\cdots a_{M-1}}$$

Now we obtain

$$\Phi(u,v,w)\,\mathscr{L}(u,v,w')=\mathscr{L}(u,v,w')\,\Phi(u,v,w)$$

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REFERENCES

1. V. A. Fateev and A. B. Zamolodchikov, Self-dual solutions of star–triangle relations in Z_N-model, *Phys. Lett. A* **92**:37–39 (1982).
2. H. Au-Yang, B. M. McCoy, J. H. Perk, S. Tang, and M. Yan, Commuting transfer matrices in the chiral Potts model: Solutions of star–triangle equations with genus >1, *Phys. Lett. A* **123**:219–223 (1987).
3. R. J. Baxter, J. H. H. Perk, and H. Au-Yang, New solution of the star–triangle relations for the chiral Potts model, *Phys. Lett. A* **128**:138–142 (1988).
4. H. Au-Yang and J. H. H. Perk, Onsager's star–triangle equation: Master key to integrability, *Adv. Stud. Pure Math.* **19**:57–94 (1989).
5. R. J. Baxter, V. V. Bazhanov, and J. H. H. Perk, Functional relations for transfer matrices of the chiral Potts model, *Int. J. Mod. Phys. B* **4**:803 (1989).
6. R. J. Baxter, Free energy of solvable chiral Potts model, *J. Stat. Phys.* **52**:639 (1988).
7. G. Albertini, B. M. McCoy, and J. H. H. Perk, Eigenvalue spectrum of superintegrable chiral Potts model, *Adv. Stud. Pure Math.* **19**:1–55 (1989).
8. R. J. Baxter, Chiral Potts model, eigenvalues of transfer matrix, *Phys. Lett. A* **146**:110–114 (1990).
9. G. Albertini, S. Dasmahapatra, and B. M. McCoy, Spectrum and completeness of the integrable 3-state Potts model: A finite size study, *Int. J. Mod. Phys. A* **7** (Suppl. 1A):1–53 (1992).
10. G. Albertini, Bethe-Ansatz type equation for the Fateev–Zamolodchikov spin model, *J. Phys: Math. Gen.* **25**:1799–1813 (1992).
11. M. Kashiwara and T. Miwa, A class of elliptic solutions to the star–triangle relations, *Nucl. Phys. B* **275**[FS17]:121–134 (1986).
12. K. Hasegawa and Y. Yamada, Algebraic derivation of broken Z_N symmetric model, *Phys. Lett. A* **146**:387–396 (1990).
13. E. Sklyanin, Some agebraic structure connected with the Yang–Baxter equation, *Funkt. Anal. Prilozheniya* **16**(4):27–34 (1982).
14. E. Sklyanin, Some algebraic structure connected with the Yang–Baxter equation, representation of quantum algebra, *Funkt. Anal. Prihozheniya* **16**(4):34–48 (1982).
15. R. J. Baxter, Eight-vertex model in lattice statistics, *Phys. Rev. Lett.* **26**:832–833 (1971).
16. R. J. Baxter, Partition function of eight-vertex model, *Ann. Phys.* (N.Y.) **70**:193–228 (1972).
17. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
18. V. V. Bazhanov, and Yu. G. Stroganov, Chiral Potts model as a descendent of the six-vertex model, *J. Stat. Phys.* **59**(3/4):799–817 (1990).
19. V. V. Bazhanov and N. Yu. Reshetikhin, Critical RSOS models and conformal field theory, *Int. J. Mod. Phys. A* **4**(1):115–142 (1989).
20. M. Jimbo, T. Miwa, and M. Okado, Solvable lattice model with broken Z_N symmetry and Hecke's indefinite form, *Nucl. Phys. B* **275**[FS17]:517–545 (1986).

21. R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain I, some fundamental eigenvectors, *Ann. Phys.* (N.Y.) **76**:1–24 (1973).
22. R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain II, equivalence to a generalized ice-type lattice model, *Ann. Phys.* (N.Y.) **76**:25–47 (1973).
23. R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain III, eigenvalues of transfer matrix and Hamiltonian, *Ann. Phys.* (N.Y.) **76**:48–71 (1973).
24. V. Pasquier, Etiology of IRF model, *Commun. Math. Phys.* **118**:357–364 (1988).
25. K. Hasegawa, Crossing symmetry in elliptic solutions of the Yang–Baxter equation and a new L-operator for Belavin's solution, *J. Phys. A* **26**:3211 (1993).
26. Y.-H. Quano, Sklyanin algebra and solvable lattice model, in *Proceedings of the Workshop on New Aspects of Quantum Field Theories*, S. Nojiri, ed. (INT-T-513, 1992), pp. 29–44.
27. K. Hasegawa, On the crossing symmetry of broken $\mathbf{Z}_N$-symmetric solutions of the Yang–Baxter equation, in *Representation Theory of Lie Groups and Lie Algebras*, T. Kawazoe, T. Oshima, and S. Sano, eds. (World Scientific, Singapore, 1992), pp. 22–58.
28. M. Takahashi and M. Suzuki, One-dimensional anisotropic Heisenberg model at finite temperature, *Prog. Theor. Phys.* **48**(6B):2187–2209 (1972).
29. F. R. Gantmacher, *Matrix Theory* (Chelsea, New York, 1959).
30. D. Mumford, *Tata Lectures on Theta I* (Birkhauser, Boston, 1983).
31. E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis*, 4th ed. (Cambridge University Press, Cambridge, 1958).